B.Sc. VI Sem "Maths" 23/4/20

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Theorem (3) मानलो (X,d) तथा (Y,ρ) दो metric Space हैं और $f: X \to Y$ एक function है तब f continuous है यदि और केवल यदि (iff)

$$\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$$, Y के प्रत्येक subset B के लिए।

Proof - Let, f is continuous and Let $B \subset Y$ then

$$f^{-1}(B) \subseteq X$$

now let, $A = f^{-1}(B)$.

$$A = f^{-1}(B) \Rightarrow f(A) \subseteq B \quad \{\because f(f^{-1}(B)) \subseteq B$$

$$\Rightarrow \overline{f(A)} \subseteq \overline{B}$$

$$\Rightarrow f(\overline{A}) \subseteq \overline{B} \quad \{\text{by } f(\overline{A}) \subseteq \overline{f(A)}$$

$$\Rightarrow \overline{A} \subseteq f^{-1}(\overline{B})$$

$$\Rightarrow \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) \quad \{\because A = f^{-1}(B)$$

Converse :-

Let, $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$, for $B \subset Y$ then we have to prove f is continuous.

Let, F is closed in Y then,

$$\overline{F} = F$$

अब परिकल्पना से हम रखते हैं:

$$\overline{f^{-1}(F)} \subseteq f^{-1}(\overline{F}) = f^{-1}(F) \quad \{\overline{F} = F$$

किंतु हम सदैव $f^{-1}(F) \subseteq \overline{f^{-1}(F)}$ रखते हैं।

परिणामत: $\overline{f^{-1}(F)} = f^{-1}(F)$

अतएव, $f^{-1}(F)$, is closed in X when ever F is closed in Y.

$\therefore$ f is continuous.

Hence Proved.

Theorem (4) :- मानलो (X, d) तथा (Y, ρ) दो metric space हैं और $f: X \to Y$ एक function है तब f continuous है यदि और केवल यदि,
$f(\bar{A}) \subseteq \overline{f(A)}$, x के प्रत्येक Subset A के लिए)

Proof :- Let, f is continuous function. and let $A \subset X$ then $f(A) \subseteq Y$ and $\overline{f(A)}$ is closed in Y.

$\because$ f is continuous and $\overline{f(A)}$ is closed in $Y \Rightarrow f^{-1}(\overline{f(A)})$ is closed in X.

now,

$$f(A) \subset \overline{f(A)}$$

$$\Rightarrow \quad A \subseteq f^{-1}(\overline{f(A)})$$

{ $\because$ x के प्रत्येक subset A के लिए,
$f^{-1}(f(A)) \supseteq A$ तथा $A \subseteq B \Rightarrow f^{-1}(A) \Rightarrow f^{-1}(B)$ }

$\Rightarrow \quad \bar{A} \subseteq \overline{f^{-1}(\overline{f(A)})}$ { $\because A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$

$\Rightarrow \quad \bar{A} \subseteq f^{-1}(\overline{f(A)})$ { $\because f^{-1}(\overline{f(A)})$ is closed

$\Rightarrow f(\bar{A}) \subseteq \overline{f(A)}$

($\because f(f^{-1}(E)) \subseteq E$, Y के किसी भी subset E के लिए)

Converse:- let ;

$f(\bar{A}) \subseteq \overline{f(A)}$ for each $A \subset X$

we have to prove,

f is continuous.

Let, F is closed in Y then $\bar{F} = F$

$\because \quad F \subseteq Y \Rightarrow f^{-1}(F) \subseteq X$

now परिकल्पना (Hypothesis) से, हम रखते हैं:

$f\left(\overline{f^{-1}(F)}\right) \subseteq \overline{f(f^{-1}(F))}$

$\Rightarrow f\left(\overline{f^{-1}(F)}\right) \subseteq \bar{F} = F \qquad \{\because \bar{F} = F$

$\Rightarrow \overline{f^{-1}(F)} \subseteq f^{-1}(F)$

किंतु हम सदैव $f^{-1}(F) \subseteq \overline{f^{-1}(F)}$ रखते हैं।

अतएव, $\overline{f^{-1}(F)} = f^{-1}(F)$

फलत: $f^{-1}(F)$; X में closed है।

$\therefore$ f is continuous

Hence Proved.