B.Sc. II Sem Maths Date: 16/4/20

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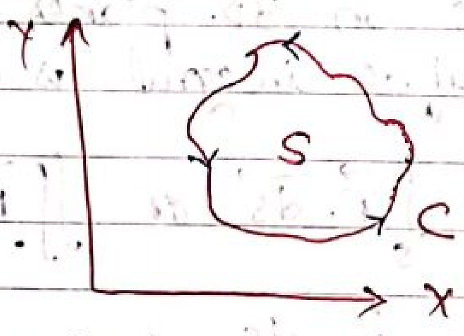
Ch- 26 Stoke's Theorem

Statement of Stoke's Theorem:– If F is any differentiable vector point fun.^{n} and S is a surface bounded by a curve C then,

$$\int_C F \cdot dr = \iint_S \text{Curl } F \cdot \hat{n} \, ds$$

where, $\hat{n}$ = unit normal vector at any point on S is drawn in the sense in which a right handed screw would move when rotated in the sense of description of C.

Remark:–



Suppose, C is the curve and S is the surface.

इस Surface S पर Curve C का direction Anti clock wise हो तो वो right handed Screw law के according तो Screw बाहर की ओर निकलेगा तो वो $\hat{n}$ होगा उसका direction बाहर की ओर होगा और ये $\hat{n}$ इस Surface S पर Perpendicular unit vector होगा

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$$\boxed{\hat{n} = \frac{\text{grad } \phi}{|\text{grad } \phi|}}$$

Note:- यदि Curve का direction clockwise है तो $\hat{n}$ पीछे की ओर होगा।

Surface Integral:- A Integral which is evaluated over the surface s is called surface integral.

Let, $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a vector point fun.^{n} and s is any surface in 3D planes.
Then, surface integral is defined as:

$$\iint_S \vec{F}.\vec{ds} \quad \text{or} \quad \int_S \vec{F}.\vec{ds} \quad \text{or} \quad \iint \vec{F}.\hat{n}\, ds$$

where $\vec{ds} = \hat{n}\, ds$
$\hat{n}$ = unit normal vector to surfaces.
or perpendicular vector over curved surface s is called $\hat{n}$.

ds - ds is the Small part of s.

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Remark: (1) If R be projection of surface S on XY plane then,

$$\iint_S \vec{F}\hat{n}\, ds = \iint_R \vec{F}\hat{n}\, \frac{dx\, dy}{|\hat{n}\cdot\hat{k}|}$$

where $ds = \dfrac{dx \cdot dy}{|\hat{n}\cdot\hat{k}|}$ [यहाँ $\hat{k}$ इसलिए आया है क्योंकि plane xy पर z (or $\hat{k}$) as a Perpendicular unit normal vector होता है।]

(2) If R be projection of surface S on YZ plane, then,

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{R_1} \vec{F}\cdot\hat{n}\, \frac{dy \cdot dz}{|\hat{n}\cdot\hat{i}|}$$

(3) If R be projection of surface S on xz-plane, then,

$$\iint_S \vec{F}\cdot\hat{n}\cdot ds = \iint_{R_2} \vec{F}\cdot\hat{n}\cdot \frac{dx \cdot dz}{|\hat{n}\cdot\hat{j}|}$$

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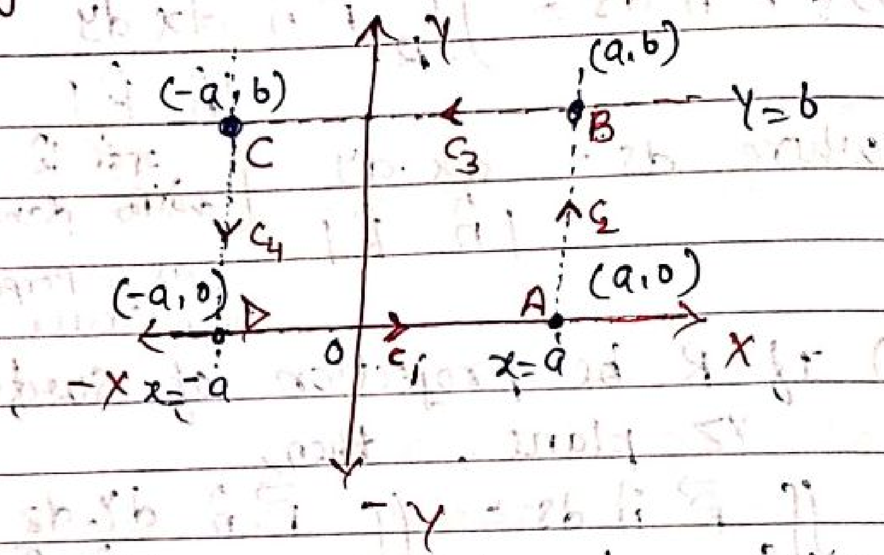
Ex:- (1) verify stoke's theorem for $\vec{F} = (x^2+y^2)i - 2xyj$ taken around the rectangle bounded by the lines $x = \pm a$, $y=0$, $y=b$.

Sol.n :-



Here 'C' is consist four sides C_1, C_2, C_3, C_4 and S is surface bounded by C.

Let,

$AB - C_2$, $BC \to C_3$; $CD \to C_4$, $DA \to C_1$

By stoke's theorem, $\int_C \vec{F}.dr = \iint_S \text{Curl}\,\vec{F}.\hat{n}.ds$ ——(1)

Since,

$$\int_C \vec{F}.dr = \int_{C_1} \vec{F}.dr + \int_{C_2} \vec{F}.dr + \int_{C_3} \vec{F}.dr + \int_{C_4} \vec{F}.dr \quad \text{——(2)}$$

Now, $\vec{F}.dr = [(x^2+y^2)\hat{i} - 2xy\,\hat{j}][dx\hat{i} + dy\hat{j}]$

$$= (x^2+y^2)dx - 2xy\,dy$$

on the line C_1: $y=0 \Rightarrow dy=0$ and,
x varies, $x=-a$ to $x=a$

on the line C_2: $x=a \Rightarrow dx=0$ and
y varies, $y=0$ to $y=b$.

on the line C_3: $y=b \Rightarrow dy=0$ and
x varies from, $x=a$ to $x=-a$

on the line C_4: $x=-a \Rightarrow dx=0$ and
y varies, $y=b$ to $y=0$

$$\because \int_C \vec{F}\cdot dr = \int_{-a}^{a} x^2\,dx + \int_0^b (-2ay)\,dy + \int_a^{-a} (x^2+b^2)\,dx + \int_b^0 2ay\,dy$$

$$= \left[\frac{x^3}{3}\right]_{-a}^{a} + \left[-2a\frac{y^2}{2}\right]_0^b + \left[\frac{x^3}{3}+b^2x\right]_a^{-a} + \left[2a\frac{y^2}{2}\right]_b^0$$

$$= \frac{1}{3}(a^3+a^3) - ab^2 + \frac{1}{3}(-a^3-a^3) + b^2(-a-a) + a(0-b^2)$$

$$\int_C \vec{F}\cdot dr = \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 \Rightarrow -4ab^2$$

——(3)

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Now curl $\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$

$$\Rightarrow \hat{i}(0) - \hat{j}(0) + \hat{k}(-2y - 2y)$$

$$\Rightarrow -4y\hat{k}$$

$$\because \iint_S \text{curl } \vec{F}.\hat{n}\, ds = \int_{x=-a}^{a} \int_{y=0}^{b} (-4y\hat{k}).\hat{k}\, dx\, dy$$

{Since S is rectangle in xy-plane so, $ds = dx\,dy$ and $\hat{n} = \hat{k}$ and limits as, $x = -a$ to $x = a$
$y = 0$ to $y = b$.}

$$\iint_S \text{curl } \vec{F}.\hat{n}\, ds = \int_{x=-a}^{a} \int_{y=0}^{b} [-4y]\, dx\, dy$$

$$= -4\int_{-a}^{a} \left[\frac{y^2}{2}\right]_0^b dx$$

$$= -\frac{4}{2}\int_{-a}^{a} (b^2 - 0)\, dx \Rightarrow -2\int_{-a}^{a} b^2 dx$$

$$= -2b^2 [x]_{-a}^{a} \Rightarrow -2b^2(a+a)$$

$$\Rightarrow -4ab^2 \text{ —— } (4)$$

Stoke's theorem is verified by eqn. (3) & (4)

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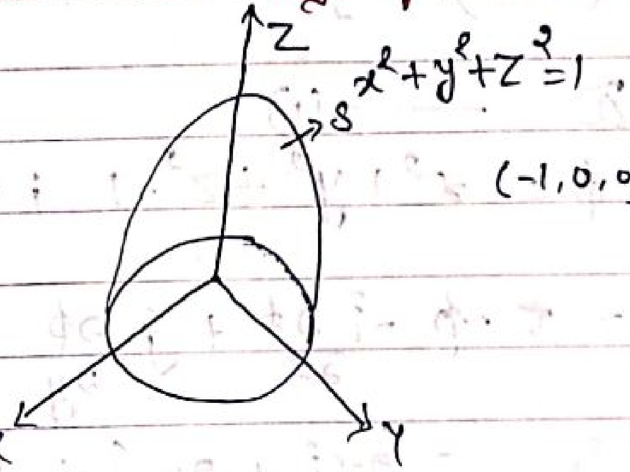
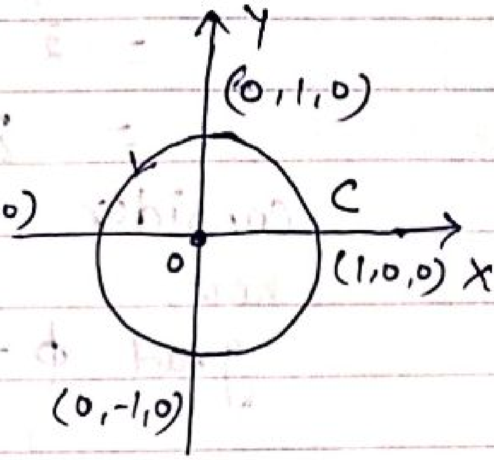
Ex. (2) Verify Stoke's theorem, for the vector field

$$\vec{F} = (2x - y)\,i - yz^2\,j - y^2 z\,k$$

over the upper half surface of $x^2+y^2+z^2=1$, bounded by its projection on the xy - plane.





Sol^n:-

Here C is the circle in XY - plane $(Z=0)$. and S is the surface of sphere $x^2+y^2+z^2=1$ in upper half,

now, $\int_c \vec{F}\cdot dr = \int_c (2x-y)\,dx - yz^2\,dy - y^2 z$

$$= \int_c (2x - y)\,dx \quad \begin{cases} Z=0 \text{ then,} \\ dz=0 \end{cases}$$

$$= \int_0^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta\,d\theta)$$

put $x = r\cos\theta = 1\cdot\cos\theta$, $Y = r\sin\theta = \sin\theta$

$dx = -\sin\theta\,d\theta$

$\theta=0$ to $\theta=2\pi$ for the curve C i.e. circle.

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$$= \int_0^{2\pi} \left[-\sin 2\theta + \left(\frac{1 - \cos 2\theta}{2} \right) \right] d\theta$$

$$= \left(\frac{\cos 2\theta}{2} \right)_0^{2\pi} + \left[\frac{1}{2} \left(\theta - \frac{\sin 2\theta}{2} \right) \right]_0^{2\pi}$$

$$= \frac{1}{2}(1-1) + \frac{1}{2}(2\pi - 0)$$

$$= \pi \quad \text{——— (1)}$$

Consider $\phi = x^2 + y^2 + z^2 - 1$ i.e. surface s.

now,

$$\text{grad } \phi = \nabla \cdot \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

Since $\hat{n} = \dfrac{\text{grad } \phi}{|\text{grad } \phi|} = \dfrac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}}$

$$\hat{n} = \frac{\cancel{2}(x\hat{i} + y\hat{j} + z\hat{k})}{\cancel{2}\sqrt{x^2 + y^2 + z^2}} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{1}}$$

$$\hat{n} = x\hat{i} + y\hat{j} + z\hat{k}$$

Let, R be the projection of s on xy plane then,

$$ds = \frac{dx \cdot dy}{|\hat{n} \cdot \hat{k}|} = \frac{dx \cdot dy}{z}$$

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$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x-y) & (-yz^2) & (-y^2z) \end{vmatrix}$$

$$\text{curl } \vec{F} = \hat{i}(-2yz + 2yz) - \hat{j}(0-0) + \hat{k}(0+1)$$

$$\text{curl } \vec{F} = \hat{k}$$

$$\therefore \iint_S \text{curl } \vec{F} \cdot \hat{n} \cdot ds = \iint_R k(x\hat{i} + y\hat{j} + z\hat{k}) \frac{dx\,dy}{z}$$

$$= \iint_R dx \cdot dy \frac{\cancel{z}}{\cancel{z}}$$

$= \text{Area of } R$ (R is circle in xy plane)

$$= \pi r^2 \Rightarrow \pi (1)^2$$

$$= \pi \quad \text{---} (2)$$

by equ. (1) and equ. (2), stoke's theorem is verified.

i.e.

$$\int_c \vec{F} \cdot dr = \iint_S \text{curl } \vec{F} \cdot \hat{n}\, ds$$

Hence Proved.

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